doi:10.1006/jabr.2000.8415, available online at http://www.idealibrary.com on IDEAL®

Polycyclic Group Rings Whose Principal Ideals Are Projective

Antonio Behn

University of Wisconsin–Madison, Madison, Wisconsin 53706
E-mail: behn@math.wisc.edu

Communicated by Efim Zelmanov

Received January 18, 2000

Let $K[G]$ be a prime group algebra with G a polycyclic-by-finite group. In this paper we prove that $K[G]$ is a CS-ring or a PP-ring if and only if either G is torsion-free or $G \cong D_\infty$ and char $K \neq 2$. The proof requires group theoretic as well as group ring arguments. © 2000 Academic Press

1. INTRODUCTION

Let R be a ring. An R-module M is said to be an essential submodule of N, written M ess N, if for every nonzero submodule $L \subseteq N$, we have $M \cap L \neq 0$. A submodule Q of M is a complement in M if Q has no proper essential extension in M. R is said to be CS if every complement ideal is a direct summand of R. In particular, every Noetherian domain is a CS-ring. Furthermore, R is said to be PP if every principal right ideal is projective. Certainly every domain has this property, and it is known that a nonsingular CS-ring is PP. Recall that the right singular ideal of a ring R is $\mathrm{Sing}(R) = \{r \in R | rQ = 0$ for some essential right ideal Q of $R\}$. R is said to be non-singular if the right singular ideal of R is zero. The main result of [JKMS00] asserts that the group algebra $K[D_\infty]$ of the infinite dihedral group D_∞ is a CS-ring if and only if char $K \neq 2$. In this paper, we classify those prime group algebras $K[G]$, with G polycyclic-by-finite, which satisfy either the CS or the PP condition. We show that this occurs if and only if either G is torsion free or $G \cong D_\infty$ and char $K \neq 2$.

Section 2 obtains information on the group theoretic structure of G when we know that $K[G]$ is prime. Section 3 contains the necessary ring theoretic

0021-8693/00 $35.00

results. Finally, in Section 4 we show by example that the main result of this paper cannot be extended to solvable-by-finite groups.

2. GROUP THEORETIC RESULTS

DEFINITION 2.2.1. Suppose $\mathscr{P}$ is a property of groups. A group G is called poly-$\mathscr{P}$ if there exists a finite chain of subgroups $1 = G_0 \lhd G_1 \lhd \cdots \lhd G_{n-1} \lhd G_n = G$ such that each of the factor groups G_i/G_{i-1} has the property $\mathscr{P}$.

In an effort to characterize those group algebras that are prime and PP, we will consider groups of the form $G = H \rtimes A$, where H is poly-infinite-cyclic, and A is finite. As we can see from the following lemma, these are essentially all the polycyclic-by-finite groups.

LEMMA 2.2.2. *The following properties of a group are equivalent*:

1. *poly*-(*cyclic or finite*)
2. *polycyclic-by-finite*
3. (*poly-infinite-cyclic*)*-by-finite*.

Proof. See [Seg83, Proposition 2]. ∎

We recall [Pas77, Theorem 4.2.10] that the group algebra $K[G]$ is prime if and only if the group G has no nonidentity finite normal subgroup. Other equivalent conditions are that $\Delta(G) = \{x \in G \mid |G : \mathbb{C}_G(x)| < \infty\}$ is torsion-free abelian or that $\Delta^+(G) = \{x \in \Delta(G) \mid o(x) < \infty\} = 1$. Both $\Delta(G)$ and $\Delta^+(G)$ are characteristic subgroups of G and will play important roles in subsequent arguments. The following lemmas will help us better understand the group structure of G.

LEMMA 2.2.3. *Let H be a torsion-free group and let A be a finite group acting on H. If $|H : \mathbb{C}_H(A)| < \infty$ then A acts trivially on H.*

Proof. Let $G = H \rtimes A$ and recall that $\Delta^+(G) = \{x \in G \mid |G : \mathbb{C}_G(x)| < \infty,\ o(x) < \infty\}$ is a characteristic subgroup of G. If $a \in A$, then

$$|G : \mathbb{C}_G(a)| \leq |G : H||H : \mathbb{C}_H(a)| < \infty.$$

Also $o(a) < \infty$ because A is finite. Thus $A \subseteq \Delta^+(G) \lhd G$. But $H \lhd G$ and, since H is torsion-free, $\Delta^+(G) \cap H = 1$. Therefore $\Delta^+(G)$ commutes with H, and in particular A acts trivially on H. ∎

LEMMA 2.2.4. *Let H be a torsion-free group and let A be a group acting on H. If the action is faithful, then $\Delta^+(H \rtimes A) = 1$.*

Proof. Let $G = H \rtimes A$ and take any element $g \in \Delta^+(G)$. Consider the group generated by this element, say $B = < g >$. Note that B is a finite group acting on H and

$$|H : \mathbb{C}_H(B)| = |H : \mathbb{C}_H(g)| = |H\mathbb{C}_G(g) : \mathbb{C}_G(g)| \leq |G : \mathbb{C}_G(g)| < \infty.$$

So, by Lemma 2.2.3, B acts trivially on H. Now write $g = ha$, where $h \in H$, $a \in A$. Since g acts trivially on H, we have $(ha)^{-1}h(ha) = h$, which yields $ha = ah$. Thus $1 = g^{o(g)} = h^{o(g)}a^{o(g)} \in H \rtimes A$. But H is torsion-free, so $h = 1$ and therefore $B \subseteq A$. Summarizing, we have $B \subseteq A$, which means that B acts faithfully on H. But B acts trivially on H, so $g = 1$ and $\Delta^+(G) = 1$. ∎

This result has particular interest when A is finite. Then $K[H \rtimes A]$ is prime if and only if A acts faithfully on H. We will need to consider the special case where $H = F$ is nilpotent. To this end, we require the following definition.

Definition 2.2.5. Let G be an arbitrary group and define $\mathscr{T}(G) \lhd G$ so that $\mathscr{T}(G)/G'$ is the full torsion subgroup of G/G'. In this way $G/\mathscr{T}(G)$ is torsion-free and abelian. Note that even when G is torsion-free, $\mathscr{T}(G)$ may be strictly larger than G'.

Notice that if $\phi(G)$ is a homomorphic image of G, then $\phi(G') = (\phi(G))'$. Therefore $\phi(\mathscr{T}(G)) \subseteq \mathscr{T}(\phi(G))$ and we have a homomorphism from $G/\mathscr{T}(G)$ onto $\phi(G)/\mathscr{T}(\phi(G))$.

Lemma 2.2.6. *Let F be a nilpotent group. If $\mathscr{Z}(F)$ is torsion-free, then $F/\mathscr{Z}(F)$ is torsion-free.*

Proof. We show, first, that $\mathscr{Z}(F)$ torsion-free implies that $\mathscr{Z}_2(F)/\mathscr{Z}(F)$ is also torsion-free. Thus suppose $x \in \mathscr{Z}_2(F)$ with $x^r \in \mathscr{Z}(F)$ for some integer $r > 0$. If y is any element of F, then since $[x, y] \in \mathscr{Z}(F)$ we have $[x, y]^r = [x^r, y] = 1$. Hence $[x, y] = 1$, since $\mathscr{Z}(F)$ is a torsion-free group. Thus we see that x is central in F and, therefore, that $\mathscr{Z}_2(F)/\mathscr{Z}(F)$ is torsion-free.

It is now trivial to prove the lemma by induction on the nilpotence class of F. Let $\overline{F} = F/\mathscr{Z}(F)$ and notice that $\mathscr{Z}(\overline{F}) \cong \mathscr{Z}_2(F)/\mathscr{Z}(F)$ is torsion-free. Thus the inductive hypothesis applies and $\overline{F}/\mathscr{Z}(\overline{F})$ is torsion-free. But then $\overline{F}$ is torsion-free and hence so is F. ∎

Lemma 2.2.7. *Let F be a torsion-free nilpotent group and let A be a finite group acting faithfully on F. Then A acts faithfully on $F/\mathscr{T}(F)$.*

Proof. We will proceed by induction on the nilpotence class of F, the result being trivial when F is abelian. Let $Z = \mathscr{Z}(F)$, $T = \mathscr{T}(F)$, $B = \mathbb{C}_A(F/T)$. We want to prove that $B = 1$. Consider $\overline{F} = F/Z$. B acts trivially on F/T; thus it also centralizes $\overline{F}/\mathcal{T}(\overline{F})$. By lemma 2.2.6, $\overline{F}$ is torsion-free, so we can apply the inductive hypothesis and conclude that B acts trivially on $\overline{F}$. Therefore $[[F, B], F] = [[B, F], F] \subseteq [Z, F] = 1$, and, by the three subgroups lemma [Isa94, Lemma 8.27], $[[F, F], B] = 1$. Thus B acts trivially on F'. If t is an arbitrary element of T, let G be the subgroup of F generated by F' and all the conjugates of t under the action of B. Since B is finite, G/F' is a finitely generated group. Also t has finite order and G/F' is abelian, so $|G : F'| < \infty$. But G is a torsion-free group, so by Lemma 2.2.3, B acts trivially on G. Since this is true for any t in T, B centralizes T.

Now let $G = F \rtimes B$, so that $T \lhd G$. Since B centralizes F/T, F/T normalizes TB/T and therefore G/T normalizes TB/T. Thus $TB \lhd G$. Note that

$$|TB : \mathbb{C}_{TB}(B)| \leq |TB : T| < \infty$$

Therefore $B \subseteq \Delta^+(TB) \lhd G$ because $\Delta^+(TB)$ is a characteristic subgroup of TB and $TB \lhd G$. Also, since F is torsion-free, $\Delta^+(TB) \cap F = 1$. So F and $\Delta^+(TB)$ are normal subgroups of G with trivial intersection. Therefore $B \subseteq \Delta^+(TB) \subseteq \mathbb{C}_G(F)$. But B acts faithfully on F, so $B = 1$. ∎

Under these circumstances we have a faithful action on a torsion-free abelian group.

LEMMA 2.2.8. *Let G be a finitely generated torsion-free Abelian group and let A be a nontrivial finite group acting faithfully on G. If A normalizes every direct summand of G, then $A \cong \mathbb{Z}/2\mathbb{Z}$ and A acts dihedrally on G.*

Proof. Write $G = G_1 \oplus G_2 \oplus \cdots \oplus G_n$ as a direct sum of cyclic groups, where $G_i =< g_i >\cong \mathbb{Z}$. Pick an element $1 \neq a \in A$. By hypothesis, a normalizes each of the direct factors G_i, so $g_i^a = g_i^{\alpha_i}$, where $\alpha_i = \pm 1$. Since the action is faithful, not all the α_i are $+1$, so we can assume $\alpha_1 = -1$.

Now write $G =< g_1 g_i > \oplus G_2 \oplus \cdots \oplus G_n$, with $i \neq 1$. Since A normalizes $< g_1 g_i >$, we get $(g_1 g_i)^a = g_1^{-1} g_i^{\alpha_i} = (g_1 g_i)^\beta$. Thus $\beta = \alpha_i = -1$ for every i and a acts dihedrally on G. If $1 \neq b \in A$, then $g^{ab} = (g^{-1})^b = g$. Thus $ab = 1$, and $A \cong \mathbb{Z}/2\mathbb{Z}$. ∎

LEMMA 2.2.9. *Let F be a torsion-free nilpotent group. If $F/\mathcal{T}(F)$ is cyclic then F is cyclic.*

Proof. We will prove the lemma by induction on the nilpotence class of F. If F is abelian, then $\mathcal{T}(F) = 1$ and there is nothing to prove. Otherwise let $\overline{F} = F/\mathcal{Z}(F)$ and note that by lemma 2.2.6, $\overline{F}$ is torsion-free. $\overline{F}/\mathcal{T}(\overline{F})$ is a homomorphic image of $F/\mathcal{T}(F)$ and it is therefore cyclic. By the inductive hypothesis this shows that $\overline{F}$ is cyclic. But $\overline{F} = F/\mathcal{Z}(F)$, so F is Abelian and hence cyclic. ∎

LEMMA 2.2.10. *Let H be a polycyclic torsion-free group. If the Fitting subgroup $F = \mathbf{F}(H)$ is cyclic, then H is cyclic.*

Proof. Let us first recall that since H satisfies the ascending chain condition on subgroups [Seh78, Proposition 2.2], it has a Fitting subgroup. Assume $H \neq 1$. Then $F \neq 1$ and hence F is infinite cyclic. Now let $1 \neq x \in H$. Since the only nontrivial automorphism of F has order 2, we know that x^2 commutes with F. But F is self-centralizing and H is torsion-free, so $1 \neq x^2 \in F$. Finally x commutes with x^2, a nonidentity element of F, so x commutes with F. Hence $x \in F$ and $F = H$. ∎

3. RING THEORETIC RESULTS

We will now try to get a better control over the PP condition. We note first that PP is inherited by subgroups.

LEMMA 3.3.1. *If $K[G]$ is PP and $H \subseteq G$ is a subgroup, then $K[H]$ is also PP.*

Proof. We can write

$$K[G] = K[H] \oplus U, \tag{1}$$

a direct sum of $K[H]$-bimodules. Now take $x \in K[H]$. We want to show that $xK[H]$ is a projective $K[H]$-module. From (1) we get

$$xK[G] = xK[H] \oplus xU.$$

But since $K[G]$ is PP by hypothesis we know that $xK[G]$ is projective and therefore a direct summand of a free $K[G]$-module. In particular $xK[H]$ is a direct summand of a free $K[H]$-module. ∎

The PP condition is equivalent to the right annihilator of any element of R being generated by an idempotent. In fact, if we let x be an arbitrary element of R, the right ideal xR is projective if and only if the short exact sequence $0 \to \mathrm{r.ann}(x) \to R \to xR \to 0$ splits. This occurs if and only if $\mathrm{r.ann}(x)$ is generated by an idempotent. We are consequently interested in understanding the idempotents in $K[G]$. Notice that from the preceding argument we cannot conclude that xR is generated by an idempotent since the splitting map $xR \to R$ is not in general the inclusion map.

LEMMA 3.3.2. *Let $G = F \rtimes A$, where F is finitely generated torsion-free nilpotent and A is finite. Then there are no nontrivial idempotents in the augmentation ideal $\omega(K[F])K[G]$.*

Proof. By [Pas77, Lemma 5.1.10], $K[G]$ is a free left K[F]-module with basis A, and therefore embeds it in $M_n(K[F])$, where $n = |A|$. This embedding sends $\omega(K[F])K[G]$ into $M_n(\omega(K[F]))$. We will proceed by induction on the nilpotence class of F to show that there are no nonzero idempotents in $M_n(\omega(K[F]))$. Let $\mathbf{e} = (e_{ij})_{ij} \in M_n(\omega(K[F]))$ be an idempotent. Consider the natural homomorphism

$$K[F] \longrightarrow K[F/\mathscr{Z}(F)]$$

which extends to

$$M_n(K[F]) \longrightarrow M_n(K[F/\mathscr{Z}(F)]).$$

Notice that $F/\mathscr{Z}(F)$ is still torsion-free and has a smaller nilpotence class than F, so since idempotents get mapped to idempotents and $\omega(K[F])$ maps to $\omega(K[F/\mathscr{Z}(F)])$, we can conclude that $\mathbf{e}$ gets mapped to $athbf0$. Therefore $e_{ij} \in I = \omega(K[\mathscr{Z}(F)])K[F]$ and $e \in M_n(I)$. Also, since $\mathbf{e}$ is an idempotent, $\mathbf{e} = \mathbf{e}^2$ belongs to any power of $M_n(I)$, so

$$e_{ij} \in \bigcap_{k=1}^{\infty} I^k = \bigcap_{k=1}^{\infty} \omega(K[\mathscr{Z}(F)])^k K[F] = (\bigcap_{k=1}^{\infty} \omega(K[\mathscr{Z}(F)])^k)K[F].$$

The latter equality occurs because $K[F]$ is a free $K[\mathscr{Z}(F)]$-module. But $K[\mathscr{Z}(F)]$ is a Noetherian domain and $\omega(K[\mathscr{Z}(F)])$ is a proper ideal, so we can apply Krull's intersection theorem to conclude that $\bigcap_{k=1}^{\infty} \omega(K[\mathscr{Z}(F)])^k = 0$. In particular $\mathbf{e} = 0$. ∎

LEMMA 3.3.3. *Let S be a ring. Assume S has a subring R such that R is a domain and both S_R and ${}_RS$ are free Noetherian R-modules. Then, for $x \in S$,* $\mathrm{r.ann}_S(x) = 0$ *if and only if* $\mathrm{l.ann}_S(x) = 0$. *In particular this result applies when $S = K[G]$, where G is polycyclic-by-finite.*

Proof. Assume $\mathrm{r.ann}_S(x) = 0$ and consider the infinite sum $R + xR + x^2R + \cdots$ of right R-modules in S. Since S_R is Noetherian, this sum can not be direct, so there is some relation of the form $x^n b_0 + x^{n+1}b_1 + \cdots + x^{n+m}b_m = 0$, where we can assume that $b_0 \neq 0$. But $\mathrm{r.ann}_S(x) = 0$, so we have $b_0 + xb_1 + \cdots + x^m b_m = 0$. Now pick $y \in \mathrm{l.ann}_S(x)$ and multiply the preceding equation on the left by y. We get $yb_0 = 0$ and since S is a free R-module, $y = 0$. The result follows by right–left symmetry.

When $S = K[G]$ and G is polycyclic-by-finite, we know there is a normal poly-$\mathbb{Z}$ subgroup $H \subseteq G$ of finite index for which $R = K[H]$ is a domain. Thus the above hypotheses are satisfied and therefore $K[G]$ has the appropriate annihilator properties. ∎

LEMMA 3.3.4. *Let $G = F \rtimes A$ where F is a finitely generated torsion-free nilpotent group, and A is finite. Assume $K[G]$ is prime and PP. Let $H \lhd F$ be a normal subgroup of F. If F/H is torsion-free Abelian, then H is normalized by the action of A, so $H \lhd G$.*

Proof. Suppose H is not normalized by the action of A, so there are some $h \in H$ and some $a \in A$ such that $h^a \notin H$. Let $u = (1-a)(1-h)^a \in K[G]$ and let e be an idempotent generating r.ann(u). Since $(1 + a + \cdots + a^{o(a)-1})u = 0$, we know $e \neq 0$ (Lemma 3.3.3). Write $e = \sum_{t \in A} t x_t$, where $x_t \in K[F]$. Since $e \in$ r.ann(u), we get the following equation

$$\begin{aligned}
0 &= ue \\
&= \sum_{t \in A} (1-a)(1-h)^a t x_t \\
&= \sum_{t \in A} (1-h)^a t x_t - \sum_{t \in A} a(1-h)^a t x_t \\
&= \sum_{t \in A} t(1-h)^{at} x_t - \sum_{t \in A} at(1-h)^{at} x_{a^{-1}at} \\
&= \sum_{t \in A} t(1-h)^{at} x_t - \sum_{t \in A} t(1-h)^{t} x_{a^{-1}t},
\end{aligned}$$

where in the last line we replaced t by at as the summation index. Thus

$$(1-h)^{at} x_t = (1-h)^t x_{a^{-1}t}, \qquad \forall t \in A.$$

Note that these equations are in $K[F]$, so we can apply the natural homomorphism $\rho_t : K[F] \longrightarrow K[F/H^t]$. Then our equation becomes

$$\rho_t(1 - h^{at})\rho_t(x_t) = 0.$$

Since $h^a \notin H$, $\rho_t(1 - h^{at}) \neq 0$. Recall that F/H^t is torsion-free Abelian; thus $K[F/H^t]$ is a domain and $\rho_t(x_t) = 0$. That is, $x_t \in \omega(K[H^t])K[F] \subseteq \omega(K[F])$. Thus $e \in w(K[F])K[G]$. By Lemma 3.3.2, $e = 0$. This is a contradiction arising from the assumption that H was not normalized by A. ∎

The following observations about an abelian group algebra will be of assistance soon:

LEMMA 3.3.5. *Let A be a finitely generated free abelian group. Then*:

1. *$K[A]$ is a UFD.*
2. *All units of $K[A]$ are of the form αg, where $\alpha \in K \setminus 0$ and $g \in A$.*
3. *If A is not cyclic, then there are $g, h \in A$ such that $1 - g - h + h^2$ is irreducible and relatively prime to $1 - g^{-1} - h^{-1} + h^{-2}$.*

Proof. We can view $K[A]$ as the polynomial ring $K[x_1, \dots, x_n]$ localized at the multiplicative set S generated by $x_1, \dots, x_n$. Since $K[x_1, \dots, x_n]$ is a UFD, so is $K[A]$. Part 2 follows from [Pas77, Lemma 13.1.9]. For Part 3, Let $u = 1 - x_1 - x_2 + x_2^2$, $v = 1 - x_1^{-1} - x_2^{-1} + x_2^{-2}$. Then we know u is prime in $K[x_1, \dots, x_n]$. Suppose $u = a/s \cdot b/t$ where $a, b \in K[x_1, \dots, x_n]$ and s, $t \in S$. Then $u|ab$ in $K[x_1, \dots, x_n]$, so $u|a$ or $u|b$. Thus either u is a unit, or u is prime in $K[A]$. From part 2 we know that u is not a unit. A similar argument holds for v. Finally, if u and v are not relatively prime, they have to differ by a unit factor, so $\alpha g(1 - x_1 - x_2 + x_2^2) = 1 - x_1^{-1} - x_2^{-1} + x_2^{-2}$ for some $f \in K - 0$ and $g \in A$. Considering the term αg on the left, we have at most four possibilities for g, but none of them make the equality possible. Thus, u and v are relatively prime. ∎

LEMMA 3.3.6. *Let F be a torsion-free nilpotent group and let $A \cong \mathbb{Z}/2\mathbb{Z}$ act dihedrally on $F/\mathcal{T}(F)$. If $K[FA]$ is PP, then F is cyclic.*

Proof. By Lemma 2.2.9, it suffices to show that $F/\mathcal{T}(F)$ is cyclic. Let $0 \neq u \in K[F]$ be arbitrary and define $w = u + au \in K[FA]$, where $1 \neq a \in A$. Note that $(1 - a)w = 0$, so, by Lemma 3.3.3, we know that the right annihilator of w is not trivial. Therefore r.ann(w) is generated by a nonzero idempotent $e = x + ay$.

Direct computation of $we = 0$ and $e^2 = e$ shows us that the following equations hold:

$$ux + u^a y = 0$$

$$yx + x^a y = y.$$

Consider now the ring epimorphism $\phi: K[F] \longrightarrow K[F/\mathcal{T}(F)]$ which commutes with the action of a. The preceding equations become

$$\overline{ux} + \overline{u}^a \overline{y} = 0 \tag{2}$$

$$\overline{yx} + \overline{x}^a \overline{y} = \overline{y}. \tag{3}$$

Suppose, by way of contradiction, that $F/\mathcal{T}(F)$ is not cyclic. By Lemma 3.3.5 there are $g, h \in F/\mathcal{T}(F)$ such that $1 - g - h + h^2$ and $1 - g^{-1} - h^{-1} + h^{-2}$ are relatively prime. Now choose u such that $\overline{u} = 1 - g - h + h^2$. Then $\overline{u}$ and $\overline{u}^a$ are relatively prime. From Eq.(1), we conclude that $\overline{u}^a$ divides $\overline{x}$, so that $\overline{x}$ is in the augmentation ideal $\omega(K[F/\mathcal{T}(F)])$, since $\overline{u}$ is in this ideal. Equation (2) shows that $\overline{y}$ is also in the augmentation ideal, and so $e \in \omega(K[F])K[G]$. But by Lemma 3.3.2, $e = 0$, which contradicts the choice of e. Thus $F/\mathcal{T}(F)$ is cyclic. ∎

We are ready for a first major step toward our goal of characterizing all polycyclic-by-finite groups whose group algebra is prime and PP.

THEOREM 3.3.7. *Let $G = F \rtimes A$, where F is a finitely generated torsion-free nilpotent group and A is a nontrivial finite group. If $K[G]$ is prime and PP then F is cyclic and $G \cong D_\infty$*

Proof. Let $T = \mathcal{T}(F)$. Since $K[G]$ is prime, A acts faithfully on F. Thus by Lemma 2.2.7, A acts faithfully on F/T, which is a finitely generated torsion-free Abelian group. Now, in order to use Lemma 2.2.8, we need to show that any direct summand of F/T is normalized by the action of A. Suppose H/T is a direct summand of F/T. Then F/H is torsion-free abelian, so, by Lemma 3.3.4, H is normalized by the action of A. Thus, by Lemma 2.2.8, we conclude that $A \cong \mathbb{Z}/2\mathbb{Z}$ and A acts dihedrally on F/T. Finally Lemma 3.3.6 shows that F is cyclic and $G \cong D_\infty$. ∎

Let us now work toward the general case.

LEMMA 3.3.8. *Let G be a polycyclic-by-finite group. Assume that $K[G]$ is PP and prime. By Lemma 2.2.2 we know that there is a poly-{infinite-cyclic} subgroup $H \lhd G$ of finite index in G. Let $F = \mathscr{F}(H)$ be the Fitting subgroup of H and let A be a finite subgroup of G. Then $K[FA]$ is also PP and prime.*

Proof. Write $C = \mathbb{C}_G(F)$ and take any $a \in A \cap C$. We will show that $a \in \Delta^+(C)$. Since F centralizes a, $\mathscr{Z}(F) \subseteq \mathbb{C}_C(a)$. Also $C \cap H = \mathbb{C}_H(F) \subseteq F$ by a property of the Fitting subgroup of a solvable group, and thus $C \cap H \subseteq \mathscr{Z}(F)$.

Now we can compute

$$|C : \mathbb{C}_C(a)| \leq |C : \mathscr{Z}(F)| \leq |C : C \cap H| = |CH : H| \leq |G : H| < \infty.$$

Therefore $a \in \Delta^+(C)$. But recall that $\Delta^+(C)$ is a characteristic subgroup of C and C is a normal subgroup of G, so $\Delta^+(C) \lhd G$. Furthermore $\Delta^+(C) \cap H = 1$ because H is torsion-free. Thus $|\Delta^+(C)| \leq |G : H| < \infty$. Since $K[G]$ is prime, it has no nontrivial finite normal subgroups, so $\Delta^+(C) = 1$. In particular $A \cap C = 1$, so A acts faithfully on F. Finally, since F is torsion-free, 2.2.4 implies that $\Delta^+(FA) = 1$ and hence $K[FA]$ is prime. ∎

Note that if A is nontrivial, then $FA \cong D_\infty$ by Theorem 3.3.7.

LEMMA 3.3.9. *The group algebra $K[D_\infty]$ is right CS if and only if* $\operatorname{char} K \neq 2$.

Proof. This is the main result of [JKMS00]. ∎

THEOREM 3.3.10. *Let $K[G]$ be prime with G polycyclic-by-finite. Then the following are equivalent*:

i. *$K[G]$ is a CS-ring.*
ii. *$K[G]$ is a PP-ring.*
iii. *G is torsion-free or $G \cong D_\infty$ and* $\operatorname{char} K \neq 2$

Proof. (i)⇒ (ii) [CH77, Theorem 5.1].

Let $x \in K[G]$. We will show that the right annihilator of x is a complement in $K[G]$. Let $W = \operatorname{r.ann}(x)$. Suppose $W \operatorname{ess} V \subseteq K[G]$ and pick an arbitrary $v \in V$. Define $I = \{r \in K[G] | vr \in W\}$. We claim that $I \operatorname{ess} K[G]$. Indeed, let X be an arbitrary nonzero right ideal of $K[G]$. If $vX = 0$, then $vX \subseteq W$, so $X \subseteq I$ and $X \cap I \neq 0$. If $vX \neq 0$, then vX is a nonzero submodule of V and, since W ess V, $vX \cap W \neq 0$. Therefore $X \cap I \neq 0$. In any case $X \cap I \neq 0$ and so I is essential in $K[G]$. Now $vI \subseteq W$, so $xvI = 0$. Thus I is an essential right ideal which annihilate xv. But $K[G]$ is semiprime Noetherian and thus non-singular [Pas77, Lemma 10.4.9]. This forces xv to be 0 so that $v \in W$. Therefore $W = V$ is a complement. Finally, since $K[G]$ is a CS-ring, W is a direct summand and is therefore generated by an idempotent. Since x was arbitrary, this shows that $K[G]$ is PP.

(ii)⇒ (iii). Suppose G is not torsion-free. By Lemma 2.2.2, there is a poly-infinite cyclic normal subgroup H of finite index in G. Let $1 \neq A \subseteq G$ be an arbitrary finite subgroup and let $F = \mathbf{F}(H)$ be the Fitting subgroup of H. By Lemma 3.3.8, $K[FA]$ is also prime and PP. Therefore, by theorem 3.3.7, F is cyclic and $A \cong \mathbb{Z}/2\mathbb{Z}$. By Lemma 2.2.10, H is cyclic. In particular $H \subseteq \Delta(G)$. Now, $\Delta(G)$ is torsion-free abelian and has a cyclic subgroup of finite index, so $\Delta(G) = \langle g \rangle$. Let $x, y \in G - \Delta(G)$, so that $g \notin \mathbb{C}_G(x)$. Now $g^x = g^{-1}$ and $g^{xy} = g$, so that $xy \in \Delta(G)$. Since x, y were arbitrary, we conclude that $|G : \Delta(G)| = 2$ and $G = \Delta(G)A \cong D_\infty$ Finally, by Lemma 3.3.9, char$K \neq 2$.

(iii)⇒ (i). If $G \cong D_\infty$ and char$K \neq 2$, then Lemma 3.3.9 shows that $K[G]$ is CS. If G is torsion-free, then $K[G]$ is a Noetherian domain [Pas89, Theorem 37.5] and therefore CS, since every nonzero right ideal is essential. ∎

4. EXAMPLES

A natural question that one could ask is, what happens if we relax the polycyclic-by-finite hypothesis on the group G—in particular, if we assume G to just be solvable-by-finite? As we will see in the following example, this extra freedom allows the possibility of different PP-rings. Let us start by considering (von Neumann) regular rings. According to a characterization due to von Neumann [Pas77, Lemma 3.1.3], a ring R is regular if and only if every finitely generated right ideal is generated by an idempotent. In particular, any regular ring is PP. In the case of group rings, it is known that $K[G]$ is regular if and only if G is locally finite and has no elements of order p if char $K = p$ [Pas77, Theorem 3.1.5]. To construct a solvable group with these properties, recall the definition of a wreath product.

DEFINITION 4.4.1. Let G and H be groups. Then the wreath product of G by H, written $G \wr H$, is defined as follows. For each $x \in H$, we let G_x be the set of ordered pairs $G_x = \{[g, x] | g \in G\}$ with multiplication defined by $[g_1, x][g_2, x] = [g_1 g_2, x]$. In this way G_x is clearly a group isomorphic to G. Moreover, if $y \in H$ then y induces an automorphism of $W = \prod_{x \in H} G_x$, the weak direct product of the groups G_x by $\prod [g_x, x]^y = \prod [g_x, xy]$. This yields an action of H on W, and we set $G \wr H = W \rtimes H$, the semidirect product of W by H.

THEOREM 4.4.2. *Consider A, a nonidentity finite abelian group, and B, an infinite periodic abelian group where A and B have no elements of order p if* char $K = p$. *Then $K[A \wr B]$ is prime and PP.*

Proof. The wreath product $A \wr B$ is solvable and has no finite normal subgroups. It is also locally finite and therefore regular. Thus the group ring $K[A \wr B]$ is prime and PP. ∎

Another interesting type of group ring to consider is $G = C_n \wr C_\infty$, the wreath product of the finite cyclic group C_n by the infinite cyclic group C_∞. Let us write $A = \prod C_n$. Then $G = A \rtimes C_\infty$ is solvable and has no finite normal subgroups, so $K[G]$ is prime. It is also helpful to assume that n is not divisible by the characteristic of K. The difficulties arise when we attempt to show that every right annihilator is generated by an idempotent. Since $K[A]$ is regular we know that all right annihilators there are generated by idempotents, but not all right annihilators in $K[G]$ lift from $K[A]$.

ACKNOWLEDGMENTS

This paper constitutes a part of the author's Ph.D. thesis written at the University of Wisconsin–Madison. The author thanks his advisor, Professor D. S. Passman, for help and encouragement throughout this project.

REFERENCES

[CH77] A. W. Chatters and C. R. Hajarnavis, *On CS-rings, Quart. J. Math. Oxford* **2**, no. 28 (1977), 61–80.

[DHSW94] N. V. Dung, D. V. Huynh, P. F. Smith, and R. Wisbauer, "*Extending Modules*," Notes in Mathematics Series 313, Pitman Res. London, 1994.

[Isa94] I. M. Isaacs, "*Algebra, a Graduate Course*," Brooks/Cole, 1994.

[JKMS00] S. K. Jain, P. Kanwar, S. Malik, and J. B. Srivastava, $K[D_\infty]$ *is a CS-algebra, Proc. Amer. Math. Soc.* **128** no. 2, (2000), 397–400.

[Pas77] D. S. Passman, "*The Algebraic Structure of Group Rings*," Wiley, New York, 1977.

[Pas89] D. S. Passman, "*Infinite Crossed Products*," Pure and Applied Mathematics, Vol. 135, Academic Press, San Diego, 1989.

[Seg83] D. Segal, "*Polycyclic Groups*," Cambridge Univ. Press, Cambridge, UK, 1983.

[Seh78] S. K. Sehgal, "*Topics in Group Rings*," Dekker, New York, 1978.